Similarity solutions for the flow into a cavity

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An investigation is made into the possible types of similarity solutions that can describe the symmetric flow of a fluid into an empty spherical cavity. The flow is homentropic, and the fluid obeys a perfect gas law $p = \kappa\rho^\gamma$. Values of γ in the range $7 \geqslant \gamma > 1$ are discussed. In this range, we find that similarity solutions in which the flow accelerates into the cavity exist for $\gamma > \frac{3}{2}$. For these solutions, the radius R of the cavity decreases as the nth power of time measured from the instant at which the cavity disappears. This power n increases monotonically as γ decreases, and attains the value 1 for $\gamma = \frac{3}{2}$. Similarity solutions in which the cavity collapses with constant velocity are given by the value $n = 1$, and such solutions are possible for all values of γ in the range considered.

1. Introduction

Similarity solutions have been the subject of investigations in many different areas of fluid dynamics. In the search for a similarity solution, one has ordinary differential equations rather than partial differential equations to discuss, which is naturally a great simplification. Moreover, similarity solutions are usually solutions of the full non-linear flow equations without the non-linear terms vanishing identically. The reduction to ordinary differential equations is a consequence of the fact that the dependent variables contain unknown functions only of some particular combination of the independent variables. In unsteady compressible flow for instance, this combination is normally r/t^n, where r is a distance variable, t is the time and n is some constant. It is clear therefore that such solutions have a singularity at the time $t = 0$, and therefore describe a rather special type of flow. One such solution is Taylor's (1950) point blast solution in which a point source of energy is instantaneously released in an otherwise undisturbed medium. This gives rise to a similarity solution. Another similarity solution is that found by Guderley (1942) in his investigation of a spherical shock wave converging on a point. The geometrical convergence of the flow causes the shock to increase steadily in strength as it approaches its centre. This flow in the final stages can be described by a similarity solution, with a singularity occurring when the shock reaches its centre.

A similar type of flow is the symmetric flow into an empty spherical cavity, which can occur in cavitation and implosion phenomena. Here too one can argue that the geometrical convergence of this flow causes it to tend to a similarity form as the flow progresses, and similarity solutions can indeed be found for this problem. The present paper is concerned with the possible forms these

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similarity solutions can have. These similarity solutions, which apply only to the flow near the cavity as the cavity radius becomes small, have not been fitted to any initial flow conditions. This is likely to be difficult, and the similarity solutions are considered in isolation. We assume that the fluid obeys a perfect gas law, and we find that two classes of similarity solutions then exist: one in which the fluid moves into the cavity with constant velocity, and the other in which the fluid accelerates into the cavity. In an earlier paper (Hunter 1960), the author found a similarity solution of this second type for the particular case of a fluid obeying a perfect gas law with $\gamma = 7$, the latter value arising from an approximate equation of state for water. In this earlier work a numerical integration of the exact partial differential equations from some given initial conditions showed that the flow near the cavity does indeed tend to the similarity form as the cavity becomes small.

There are dimensional reasons for expecting the flow near the cavity to tend to a similarity form as the cavity gets small. The relevant length scale for the flow in this region is the cavity radius R. Any dimensional length scale eventually becomes too large to be relevant as the collapse progresses. One would expect the velocity scale for the flow to be that of the cavity wall. This velocity we shall denote by $\dot{R}$, where the dot denotes differentiation with respect to time t. If inward flow velocities tend to become very large as $R \to 0$, then $\dot{R}$ is indeed the only velocity large enough to provide a scale. However, it is also possible for the cavity to collapse finally with a constant velocity. Using $\dot{R}$ as the relevant velocity scale, one can write an expression for the radial flow velocity as

$$u = \dot{R} f_1\left(\frac{r}{R}, \frac{t}{T}\right), \tag{1}$$

where r is radial distance from the centre of the cavity, T is some dimensional time scale, and f_1 is an unknown function. The function f_1 could also depend on any dimensionless parameters there might be. Suppose that time is measured from the instant at which $R = 0$. Then, for sufficiently small times before R becomes zero, we might expect to be able to approximate to (1) by

$$u = \dot{R} f_1\left(\frac{r}{R}, 0\right) = \dot{R} f\left(\frac{r}{R}\right), \quad \text{say.} \tag{2}$$

It should be noted that this expression for u satisfies the kinematic boundary condition that the fluid velocity should be equal to that of the cavity on the cavity wall $r = R$ provided $f(1) = 1$.

An equation of state for the fluid is needed. We shall suppose that the flow is homentropic, and that the pressure p and the density ρ of the fluid at all points are connected by an equation of the perfect gas type

$$p = \kappa \rho^{\gamma}, \tag{3}$$

where κ and γ are constants. Using this equation of state the equations of conservation of momentum and mass for radial flow can be written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\gamma - 1}\frac{\partial c^2}{\partial r} = 0, \tag{4}$$

$$\frac{\partial c^2}{\partial t} + u\frac{\partial c^2}{\partial r} + (\gamma - 1)\, c^2\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0, \tag{5}$$

where we have used $c^2 = dp/d\rho$, the square of the velocity of sound in the fluid, as the single thermodynamic variable necessary to describe the state of the fluid.

An expression for c^2 corresponding to the expression (2) for u is

$$c^2 = \dot{R}^2 g\left(\frac{r}{R}\right), \tag{6}$$

where g is some unknown function. One can argue for such an expression in the same way as before. The flow is strongly influenced by the fact that there is a vacuum inside the cavity, and so pressure and therefore also c^2 must vanish on the cavity wall. This boundary condition of $c^2 = 0$ therefore does not furnish a dimensional scale for c^2, and so we use the only velocity squared scale provided by the boundary conditions at the cavity, which is $\dot{R}^2$. Notice that the condition $c^2 = 0$ at the cavity is satisfied if $g(1) = 0$.

Solutions of the flow equations (4) and (5) with the similarity forms (2) and (6) are possible. The two partial differential equations of flow become two ordinary differential equations to be solved for the functions f and g, provided only that

$$\frac{R\ddot{R}}{\dot{R}^2} = \text{const.} \tag{7}$$

This latter equation is satisfied by

$$R = A(-t)^n, \tag{8}$$

where A and n are constants. The only other possible solution of (7) is for R to be an exponential function of time, but we do not consider this possibility since it means that the cavity does not collapse in a finite time. We shall also limit our discussion to cavities which collapse with a non-zero velocity, so we shall suppose $0 < n \leqslant 1$. When $n = 1$, the cavity has a constant velocity of collapse. For $0 < n < 1$, the cavity accelerates inwards always and its final velocity at the instant $t = 0$ is infinite.

The index n appears in the ordinary differential equations one obtains for the similarity solution functions f and g. The cavity boundary conditions $f(1) = 1$, $g(1) = 0$ define a solution of these ordinary differential equations once γ and n are given, but most of these solutions are not acceptable. This is due to the fact that most of the solutions have singularities for some value of r/R which corresponds to the so-called limiting characteristic curve. This is the inward propagating characteristic which reaches the origin $r = 0$ at the instant when the cavity completes its collapse. Because such singularities must exist for all time, and so must have been fed into the flow initially at precisely the right point to ensure their propagating to $r = 0$ at exactly the instant at which R becomes zero, we reject all solutions with such singularities as unrealistic. In the case $\gamma = 7$ investigated earlier, it was found that a solution without a singularity on the limiting characteristic is possible for the particular value $n = 0{\cdot}5552$. In the present work we look for acceptable solutions for smaller values of γ. Naturally the range $1 < \gamma \leqslant \frac{5}{3}$ is of particular interest, since it is relevant to perfect gases. We find that similarity solutions with accelerating fronts, that is with $0 < n < 1$, are possible for $7 \geqslant \gamma > \frac{3}{2}$. For the solutions that have been found the value of

n increases as γ decreases, and reaches the value 1 for $\gamma = \frac{3}{2}$. Thus, as γ decreases, the collapse of the cavity becomes progressively less violent.

A rough explanation based on energy considerations can be given to account for the difference in behaviour for different values of γ. The collapse of the cavity is caused by a pressure difference between the cavity and the surrounding fluid. The pressure forces push the fluid into the cavity and, in so doing, do work on the fluid. This work is converted either into kinetic energy of motion or into compressional energy. In the extreme case of an incompressible fluid, which can be regarded as the case $\gamma = \infty$, all the work done must become kinetic energy. Actually, for an incompressible fluid, the similarity form (2) is an exact consequence of the continuity equation, and it can easily be seen that the asymptotic formula for the cavity radius as $R \to 0$ is given by (8) with $n = \frac{2}{5}$. For the finite values of γ considered, n is larger than $\frac{2}{5}$, and its increasing as γ decreases represents the fact that the collapsing motion is less violent when the fluid is more easily compressible, and more of the available energy is converted into compressional energy rather than into kinetic energy. It may be that for sufficiently small γ it is impossible to have the fluid accelerating into the cavity in the final part of the collapse, but only to have it moving in with a constant velocity. This is the case as far as the similarity solutions so far discovered are concerned, though it has not been established generally. The similarity solutions possible in the case $n = 1$ when the cavity collapses with a constant velocity are discussed in § 5.

One result of the analysis is that both similarity solutions with accelerating fronts and solutions with constant velocity fronts are possible for $\gamma > \frac{3}{2}$. In this range therefore, assuming that the similarity solutions are indeed the correct asymptotic forms for describing the flow near the collapse point, there is a choice of possible asymptotic forms for flows. The asymptotic form attained by a particular flow will depend on its initial conditions. Clearly any of the similarity solutions can be achieved if the initial conditions are so chosen that the similarity solution concerned is the exact solution to the flow problem. We should therefore expect there to be a class of initial conditions which leads to accelerating solutions, and another class which leads to constant-velocity solutions. The energy argument given above suggests that the accelerating solutions can be obtained if there is a sufficiently strong pressure gradient pushing fluid into the cavity. It is possible, too, that one of the types of similarity solution is unstable whereas the other is not, so that the latter tends to occur. If we begin to consider the stability of solutions, however, we must also recognize the possibility that the radially symmetric similarity solutions are unstable with respect to non-radially symmetric disturbances. This is certainly the case for an incompressible fluid, as Birkhoff (1954) has shown that the accelerating solution which exists in this case is unstable.

2. The similarity equations

As mentioned in § 1, the assumption of the similarity forms (2) and (6), where R satisfies equation (8), reduces the partial differential equations of flow to ordinary differential equations, which we shall now consider. For mathematical convenience we shall not use r/R as the similarity variable, but the variable ξ defined by

$$\xi = -\left(\frac{R}{r}\right)^{1/n} = \frac{A^{1/n}t}{r^{1/n}}. \tag{9}$$

From expression (8) for the cavity radius we can derive the relation

$$\dot{R} = -nA^{1/n}R^{1-(1/n)}, \tag{10}$$

from which it is apparent that the similarity expressions (2) and (6) can be written in the equivalent forms

$$u = -nA^{1/n}r^{1-(1/n)}F(\xi), \quad c^2 = n^2A^{2/n}r^{2-(2/n)}G(\xi), \tag{11}$$

where F and G are functions to be determined. At the cavity wall $\xi = -1$, and the boundary conditions there give

$$F(-1) = 1, \quad G(-1) = 0. \tag{12}$$

Substitution of (11) into the flow equations (4) and (5) yields two ordinary differential equations

$$(\gamma-1)(1+\xi F)F' + \xi G' + (1-n)[(\gamma-1)F^2 + 2G] = 0, \tag{13}$$

$$(\gamma-1)\xi GF' + (1+\xi F)G' + (1+n+\gamma-3n\gamma)FG = 0. \tag{14}$$

Since we are seeking a solution for the flow outside the cavity before the collapse, we shall be interested in all $r \geqslant R$, which corresponds to $-1 \leqslant \xi \leqslant 0$. To solve for F and G in this range, we have the boundary conditions (12) at $\xi = -1$ which will completely determine the solution once γ and n are specified. However, not all these solutions are acceptable since it can be shown that two further conditions must be imposed on F and G. We shall suppose $n \neq 1$ for the time being.

First, we expect the collapsing flow to become small at large distances from the cavity, so that $u \to 0$ and c tends to some finite value as $r \to \infty$. The arguments given above for the existence of a similarity solution refer only to the region near the cavity, and we expect the similarity solution to apply only in a limited region. At the outer edge of this region, it must be matched to a solution in which u and c remain finite. By (11),

$$u = -\frac{nA(-\xi)^{1-n}F(\xi)}{(-t)^{1-n}}, \quad c^2 = \frac{n^2A^2(-\xi)^{2-2n}G(\xi)}{(-t)^{2-2n}}. \tag{15}$$

Now the value $\xi = 0$ refers to all $r > 0$ at $t = 0$. Thus, if u and c^2 are to remain finite for finite r as $(-t) \to 0$, we must have

$$(-\xi)^{1-n}F(\xi) = (-\xi)^{2-2n}G(\xi) = 0 \quad \text{at} \quad \xi = 0. \tag{16}$$

Another and more significant condition is that obtained from a consideration of the singular points of equations (13) and (14). These equations can be written as

$$(\gamma-1)[(1+\xi F)^2 - G\xi^2]F' = (3n\gamma-n-\gamma-1)\xi FG - (1-n)(1+\xi F)[(\gamma-1)F^2 + 2G], \tag{17}$$

$$[(1+\xi F)^2 - G\xi^2]G' = 2(1-n)\xi G^2 + (3n\gamma-n-\gamma-1)FG + 2(n\gamma-1)\xi GF^2. \tag{18}$$

From this it is apparent that they have singular points where $[(1+\xi F)^2 - G\xi^2]$ vanishes. The cavity boundary conditions (13) at $\xi = -1$ make this expression vanish, so that $\xi = -1$ is a singular point. There are actually two possible solutions satisfying these boundary conditions at $\xi = -1$. One has $G = 0$ everywhere, which we reject for the obvious reason that it has zero density everywhere, and the other has a unique regular power series expansion about $\xi = -1$. This latter solution is the one we must accept. For this solution, $[(1+\xi F)^2 - G\xi^2]$ becomes negative as ξ increases from -1, but must be positive and equal to unity at $\xi = 0$ by condition (16). There must therefore be another singular point of our equations in the range $-1 < \xi < 0$ with which we are concerned. Let us denote this point by $\xi = \xi_0$.

This situation is well known in similarity solution problems, and was first encountered by Guderley (1942) in his investigation of the similarity solution for a converging spherical shock wave. It can easily be seen that the curve $\xi = \xi_0$ in the (r, t)-plane is a characteristic of the flow. In particular, in our problem, it is the inward propagating characteristic which reaches the origin O at time $t = 0$. Now we expect F and G to be regular functions of ξ at the singular point $\xi = \xi_0$. If they were irregular, then these irregularities would persist in the flow for all time and propagate along the ingoing characteristic. This could happen only if they were fed into the flow initially at exactly the right position for them to reach the centre of the flow at the instant when the collapse is completed. Such a situation is far too special to be of interest to us, and we therefore demand that F and G are regular throughout $-1 < \xi \leqslant 0$. It is then necessary for the right-hand sides of (17) and (18) to vanish at the singular value $\xi = \xi_0$ and this condition is not normally satisfied by the solution which satisfies the cavity boundary conditions (12). However, n is still a free parameter, and we shall investigate how it can be chosen, once γ is specified, so that a solution of equations (13) and (14) for F and G satisfying all the necessary conditions can be obtained.

It is a special feature of this problem that the cavity surface $\xi = -1$, as well as the limiting characteristic $\xi = \xi_0$, gives rise to a singular point of the similarity solution equations. This is a consequence of the fact that $c = 0$ on the cavity surface, and so the characteristic condition $dr/dt = u \pm c$ becomes $dr/dt = u$, which is satisfied by virtue of the cavity surface being a material surface.

3. Investigation of the similarity solutions

The solutions of (13) and (14) are most conveniently investigated by means of a change of variables. If we define

$$Y = -\xi F, \quad Z = \xi^2 G, \tag{19}$$

then equations (13) and (14) show that

$$\begin{aligned} dY : dZ : d\xi/\xi = {} & (\gamma-1)\, Y(Y-1)(nY-1) - Z[3(\gamma-1)\, nY + 2n - 2] : \\ & (\gamma-1)\, Z[-2nZ + 2n\gamma Y^2 + Y(\gamma - 3 + n - 3n\gamma) + 2] : (\gamma-1)\,[(Y-1)^2 - Z]. \end{aligned} \tag{20}$$

The first ratio gives an equation involving Y and Z only, and our problem can be regarded as that of finding a suitable solution of it. Thus we have

$$\frac{dY}{dZ} = \frac{(\gamma-1)\, Y(Y-1)(nY-1) - Z[3(\gamma-1)\, nY + 2n - 2]}{(\gamma-1)\, Z[-2nZ + 2n\gamma Y^2 + Y(\gamma - 3 + n - 3n\gamma) + 2]}. \tag{21}$$

The cavity boundary conditions (12) mean that the required solution of (21) must pass through the point $Y = 1$, $Z = 0$. This point, which we shall denote by C, is a saddle-point singularity of (21) for $\gamma > 1$ and $n \neq 1$. Only two solution curves therefore pass through C. One of these solutions is the line $Z = 0$. This is the solution with zero density which we met with and rejected earlier. The other solution through C is a curve tangent to the line

$$Z = \frac{\gamma(\gamma-1)(1-n)(Y-1)}{(2+n-3n\gamma)} \tag{22}$$

at C. This solution corresponds to the regular solution of the equations for F and G mentioned earlier. It is clear that we shall be concerned only with the region $Z \geqslant 0$, since $Z < 0$ corresponds to negative values of c^2. We therefore follow our solution curve out from C into the region $Z > 0$.

The conditions (16) on the solution for F and G at $\xi = 0$ require that our solution has to go through the origin O of the (Y, Z)-plane, so that this point corresponds to $\xi = 0$. The origin is a nodal singularity of (21) into which integral curves are drawn, although in fact all functions have regular expansions about this point.

The complicated part of finding a suitable integral curve going from C to O is associated with the singular value $\xi = \xi_0$ at which $(1+\xi F)^2 - \xi^2 G = 0$. This latter relation can be written as $(Y-1)^2 - Z = 0$, and so defines a parabola in the (Y, Z)-plane. The situation in the (Y, Z)-plane is that our integral curve from C initially goes above this parabola and so must cross it to reach O. We saw earlier that F and G must be regular at $\xi = \xi_0$ and this means that Y and Z must be regular functions of ξ at the point where the integral curve in the (Y, Z)-plane crosses the critical parabola $Z = (Y-1)^2$. It follows from (20) that this is only possible if both the numerator and denominator of (21) vanish at this crossing-point.

Now the points at which both the numerator and denominator of (21) vanish are the singular points of (21). As we have already seen, two of these are O and C, the latter lying on the critical parabola, though our solution must clearly cross the critical parabola at another point. There are four other singular points. One we shall denote by A at

$$Y = 1/n, \quad Z = 0;$$

the point B at

$$Y = 2/n(3\gamma-1), \quad Z = 3(\gamma-1)^2/n^2(3\gamma-1)^2;$$

and the two points on the critical parabola given by the roots of the quadratic

$$2(\gamma-1)\,nY^2 + (5n-3+\gamma-3n\gamma)\,Y + 2(1-n) = 0, \tag{23}$$

which may or may not be real. We shall denote these points by D and E, with E corresponding to the large root. Clearly therefore our regularity condition requires that our solution curve cross the critical parabola at D or E only. (The point B lies on the critical parabola only for special values of γ and n and, when it does, it coincides with either D or E. The point A lies on the critical parabola only for the special case $n = 1$ which is investigated separately below.)

It can also be seen from (20) that if our solution crosses the critical parabola at a point other than D or E, then $d\xi/dY = d\xi/dZ = 0$ and so ξ attains a turning value. The functions Y and Z are not then single valued functions of ξ and so are unacceptable.

However, the condition that our solution curve cross the critical parabola at D or E is only a necessary and not a sufficient condition for Y and Z to be regular functions of ξ at the crossing point. Provided $dY/d\xi \neq 0$ at the crossing point, then $Z = Z(Y)$ must also be a regular function. We can investigate the behaviour of Z as a function of Y at D or E directly from equation (21). Suppose that $Y = Y_0$ and $Z = Z_0$ at the singular point under consideration, and write $y = Y - Y_0$, $z = Z - Z_0$. Then equation (21) becomes

$$\frac{dy}{dz} = \frac{az + by + \text{quadratic and higher-order terms}}{cz + dy + \text{quadratic and higher-order terms}}, \tag{24}$$

where

$$\left.\begin{aligned} a &= 2 - 2n - 3n(\gamma - 1)Y_0, \quad b = (\gamma - 1)[4nY_0 - 3n - 2Y_0 + 1], \\ c &= -2n(\gamma - 1)Z_0, \quad d = (\gamma - 1)Z_0[4n\gamma Y_0 + \gamma - 3 + n - 3n\gamma]. \end{aligned}\right\} \tag{25}$$

The study of differential equations of this type is classical. It is well known that there are no regular integrals for $z = z(y)$ if $(b-c)^2 + 4ad < 0$. In this case the solutions all spiral into the singular point, or else circle it in the special case of $b + c = 0$. For $(b-c)^2 + 4ad > 0$ and $ad - bc > 0$, the singular point is a saddle point, and the two solutions which actually go into the singular point represent regular expansions $z = z(y)$. In the case of $(b-c)^2 + 4ad > 0$ and $ad - bc < 0$, the singular point is a node and the situation is a little more complicated. The form of the solutions curves near the singular point is as shown in figure 1. Analytically the curves are described by the equation

$$az + (\lambda_2 - c)\,y = k[az + (\lambda_1 - c)\,y]^{\lambda_2/\lambda_1}, \tag{26}$$

where the constant k varies for the different curves, and λ_1 and λ_2 are the roots of the quadratic

$$\lambda^2 - \lambda(b+c) + (bc - ad) = 0. \tag{27}$$

We choose λ_2 to be the larger in absolute magnitude. We shall refer to the solution given by $az + (\lambda_1 - c)\,y = 0$ (i.e. $k = \infty$) as the minor-axis solution, and that given by $az + (\lambda_2 - c)\,y = 0$ (i.e. $k = 0$) as the major-axis solution. As can be seen from figure 1, all the other solutions are tangent to the major-axis solution.

In general both the major- and minor-axis solutions represent regular expansions for $z = z(y)$, and therefore for $Z = Z(Y)$. The only exceptions arise when the ratio λ_2/λ_1 is a positive integer. This can be seen by substituting a series expansion

$$z = \beta_1 y + \beta_2 y^2 + \beta_3 y^3 + \dots, \tag{28}$$

into equation (24). Solving for the coefficients, we get a quadratic for β_1,

$$a\beta_1^2 + \beta_1(b-c) - d = 0, \tag{29}$$

the two roots of which can be written as $\beta_1 = (c - \lambda_2)/a$ and $\beta_1 = (c - \lambda_1)/a$. These two values give rise to the major- and minor-axis solutions respectively. For the later coefficients in the series, we have equations of the form

$$\beta_m[mb + (m+1)\,a\beta_1 - c] = \text{terms involving } \beta_1, \beta_2, \dots, \beta_{m-1}. \tag{30}$$

For $\beta_1 = (c-\lambda_1)/a$, the term in the square bracket on the left-hand side of (30) is $[m\lambda_2-\lambda_1]$, and for $\beta_1 = (c-\lambda_2)/a$, it is $[m\lambda_1-\lambda_2]$. There is never any difficulty in solving for all the β's in the former minor-axis case, nor indeed in the latter major-axis case except when $\lambda_2/\lambda_1 = N$, where N is a positive integer $\geqslant 2$. Then equation (30) for the case $m = N$ degenerates into a relation connecting the already determined $\beta_1, \beta_2, \ldots, \beta_{m-1}$, which in general is not satisfied. In this case we no longer have a regular expansion $z = z(y)$ since logarithmic terms must be included in the series.

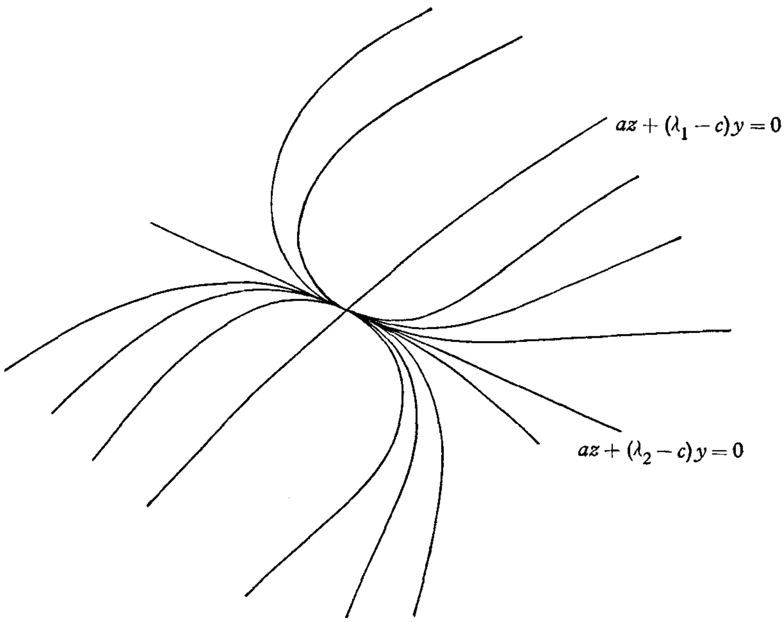

FIGURE 1. The integral curves in the neighbourhood of a node.

The one remaining possibility to be discussed is the case $(b-c)^2+4ad = 0$. The singular point is again a node, though it is special in that the major- and minor-axis solutions coincide. Equation (27) for λ has equal roots. There is no difficulty in evaluating the coefficients of our power series (28), so that the two coincident axis solutions represent a regular expansion.

Our mathematical problem in the (Y, Z)-plane is now finally clear. We must look for cases in which a regular solution through D or E links up with our solution from the saddle-point C. After crossing the critical parabola at D or E, our solution must go into O. The investigation of the solution curves in the (Y, Z)-plane has to be carried out numerically, since I am unable to integrate equation (21) analytically except in a few special cases.

4. Integrations of the (Y, Z)-equation

We have already restricted our range of interest in the (γ, n)-plane to the region $0 < n < 1$ and $1 < \gamma \leqslant 7$, and we shall now find further restrictions. One condition we shall impose is that

$$n > \frac{2}{5-3\gamma^{-1}}. \tag{31}$$

It was shown in the earlier paper (Hunter 1960) that there is an infinite amount of energy concentrated at the collapse point $r = 0$, $t = 0$ if inequality (31) is not obeyed.

Another important restriction comes from the fact that D and E do not exist unless either

$$n \geqslant \frac{(3\gamma^2 - 6\gamma + 7) + 2[2(\gamma - 1)]^{\frac{3}{2}}}{9\gamma^2 - 14\gamma + 9}, \tag{32}$$

or

$$n \leqslant \frac{(3\gamma^2 - 6\gamma + 7) - 2[2(\gamma - 1)]^{\frac{3}{2}}}{9\gamma^2 - 14\gamma + 9}, \tag{33}$$

and so it is clear from our earlier discussion that one of these inequalities must hold. Actually the second possibility is excluded by condition (31) except in the range $1 < \gamma < 1{\cdot}053$. Except in this region, we need use only inequality (32) since it implies (31) for $1 < \gamma \leqslant 7$. When (32) holds and $n < 1$, then both D and E lie on the critical parabola in the range $0 < Y < 1$, and provide possible crossing places for a curve going from C to O.

The numerical calculations were so organized that the value of γ was first fixed, and then integrations were carried out for various values of $n < 1$ satisfying inequality (32). The singular points D and E appear and coincide when (32) is an inequality, and lie on the critical parabola in the range $0 < Y < 1$. As n increases, they separate with E moving down the parabola towards C, and D moving up it towards $Y = 0$, $Z = 1$. Of the other singular points, A is always to the right of the region of interest, while B lies in the range $0 < Y < 1$. Actually it is found that there are only two possibilities as far as the curve from C is concerned. One arises when

$$1 < \gamma < \tfrac{5}{3} \quad \text{and} \quad \frac{2 + 3^{\frac{1}{2}}(\gamma - 1)}{3\gamma - 1} > n. \tag{34}$$

In this case E is a saddle and the point B is a node interposed between C and E. The curve from C goes into B before it can cross the critical parabola. An investigation of equations (20) shows that the value $\xi = 0$ is attained at B, whereas our condition (16) requires that it should be attained at O. All integral curves through B therefore are inadmissible.

The second possibility as far as the curve from C is concerned is that it goes to E. This happens when E is either a spiral or a node. Only in the latter case are regular curves through E possible, and the point B does not then lie between C and E.

To examine things in detail, the two regular curves that go through E when it is a node were integrated in the direction of increasing Y, and the solution curve through C was integrated in the direction of decreasing Y. We looked for cases in which either of the two regular curves from E join up with that from C. A Runge–Kutta integration method was used, with four term Taylor series expansions to start the integrations. We shall now summarize the results.

(i) *The range* $7 \geqslant \gamma > 2{\cdot}4$

In this range, the results are similar to those found in the earlier work for $\gamma = 7$. The point E is either a node or a spiral, and for given γ, a unique value of n is found for which the minor-axis solution from E links up with the solution from C.

When E first appears, it is a node and its minor-axis solution goes above C. As n increases, this solution gradually drops down and actually passes through C for one unique value. Figure 2 shows an example of this, together with other integral curves in the (Y, Z)-plane. For larger values of n, the minor-axis solution curls round under itself for some value of Y less than 1 and heads for O. As n is still further increased, the singular point E becomes a spiral and no regular solutions through E are possible. The values of n for which these different events occur are shown in table 1.

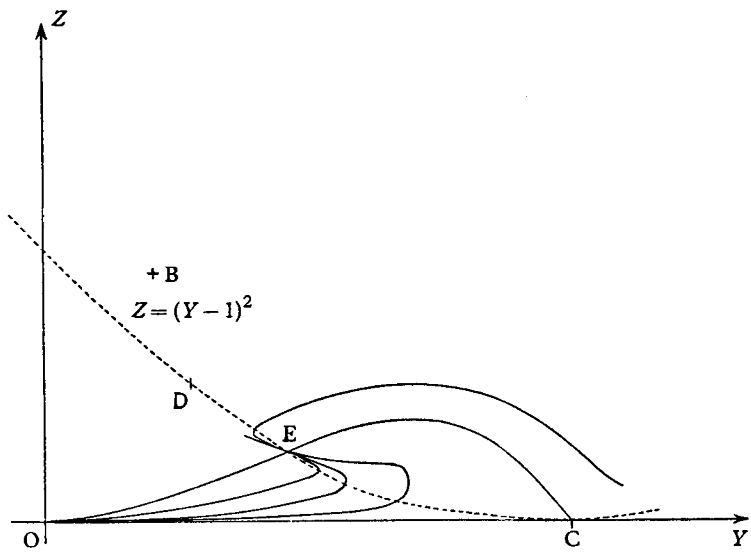

FIGURE 2. Integral curves in the (Y, Z)-plane when the minor-axis solution through E links up with the solution through C.

γ	E appears	Regular solution possible	E changes to a spiral
7	$n = 0{\cdot}5544$	$n = 0{\cdot}5552$	$n = 0{\cdot}5642$
5	0·5930	0·6009	0·6104
3	0·6667	0·7086	0·7114
2·5	0·6989	0·7641	0·7643
2·45	0·7026	0·7710	0·7711

TABLE 1

One can also verify that in the cases where the solution from C links up with the minor-axis solution through E, it goes on into O and not to B. We therefore have similarity solutions which satisfy all the required conditions. There is no difficulty to going on and calculating Y and Z as regular functions of ξ $(dY/d\xi \neq 0)$ and then the functions F and G. This has not been done though, since we are restricting our investigation to establishing the existence of these similarity solutions.

One can see in table 1 that, as γ decreases, the value of n for which a regular solution is possible gets very close to the value of n at which E becomes a spiral. Actually the two values coincide for some value of γ just in excess of 2·4. For this value of γ, the minor-axis solution from E links up with the solution from C at the particular value of n when $(b-c)^2+4ad = 0$. This is the special case discussed in § 3 when the major- and minor-axis solutions coincide. For smaller values of γ, the minor-axis solution from E always passes above C when E is a node, but we can have the major-axis solution from E linking up with the curve through C. The value $\gamma = 2{\cdot}4$ therefore marks a smooth transition to a different type of mathematical behaviour, though there does not appear to be any physical significance to this transition.

In the range $7 \geqslant \gamma > 2{\cdot}4$ no cases were found in which a major-axis solution through E links up with the curve from C. However, the accurate determination of this major-axis solution is more difficult than that of the minor-axis solution. This aspect of the problem is discussed more fully below in the next subsection.

(ii) *The range* $2{\cdot}4 \geqslant \gamma \geqslant \frac{3}{2}$

In this region, values of n have been found for which the major-axis solution from E links up with the solution through C. The set of values for which a regular solution is possible in this way are given in table 2. Note that the value of n

γ	E appears	Regular solution possible	E changes to a spiral
2·4	0·7066	0·7782	0·7782
2·3	0·7148	0·7937	0·7939
2·0	0·7445	0·8502	0·8568
1·9	0·7560	0·8735	0·8865
1·8	0·7690	0·8996	0·9244
1·7	0·7834	0·9290	0·9770
1·6	0·7997	0·9623	>1
1·55	0·8085	0·9806	>1
1·5	0·8182	1·0000	>1

TABLE 2

for which E becomes a spiral is 1 for $\gamma = \frac{5}{3}$, and becomes greater than 1 for smaller γ, and so this transition no longer concerns us. There is also the possibility of E being a saddle for $\gamma < \frac{5}{3}$ as mentioned above, though this does not give rise to acceptable solutions.

The fact that the value of n for which a regular solution is possible attains the value 1 for $\gamma = \frac{3}{2}$ can be checked theoretically. To discuss values of n near 1, we write $n = 1-\delta$ with δ small. The co-ordinates of E are approximately $Y = 1-\frac{1}{2}\delta$, $Z = \frac{1}{4}\delta^2$. To study the behaviour of curves in the region of C and E, we introduce scaled variables y' and z', where $Y = 1+\delta y'$, $Z = \delta^2 z'$. The co-ordinates of C are $y' = 0$, $z' = 0$, those of E are $y' = -\frac{1}{2}$, $z' = \frac{1}{4}$ and the (Y, Z)-equation is

$$\frac{dy'}{dz'} = \frac{y'(y'-1)-3z'}{(\gamma-1)\,z'(1+2y')}, \tag{35}$$

when small terms of order δ are neglected. This approximation is valid in the region near E and C. One can check that this equation has C as a saddle-point singularity and E as a node for $\gamma < \frac{5}{3}$, as they are of the full (Y, Z)-equation.

Now when $\gamma = \frac{3}{2}$, the solution of (35) through C is $2z'+y' = 0$, which is also the major-axis solution through E, so that we have the correct type of link-up.

The regular solutions calculated in the (Y, Z)-plane keep closer to the Y-axis as γ decreases and n increases, and tend to the line $Z = 0$ as the solution when $\gamma = \frac{3}{2}$ and $n = 1$.

Actually the behaviour of the major-axis solutions is much more varied than that of the minor-axis solutions. The reason for this is that it is possible for the major-axis solution from E to change from going above the curve from C to going below it in two ways. Either it does so smoothly with the two curves linking up at some stage, or else this change takes place discontinuously with no link-up occurring. This latter situation can occur when we approach one of the special cases discussed in § 3 of λ_2/λ_1 equal to a positive integer. We can see clearly what happens if we consider the simple model equation

$$\frac{dy}{dz} = \frac{y}{mz+By^2}, \tag{36}$$

where m and B are constants. The two roots of the equation for λ are now $\lambda = m$ and $\lambda = 1$, and we shall be interested in what happens as the value of m varies around the integral value 2. The point $y = 0$, $z = 0$ is a node, and equation (36) can be integrated easily to give as the major-axis solution

$$z = By^2/(2-m). \tag{37}$$

This expression is not valid for $m = 2$, when the integral becomes

$$z = By^2 \log y, \tag{38}$$

which is not regular. If we consider the intercept of the curve (37) with the line $y = 1$ for example, we see that it changes from being large and positive to large and negative as m goes through the value 2.

Naturally the details of such a transition can be more complicated when we have a more complicated equation than (36) such as (21). Equation (36) is linear for z as a function of y, whereas (21) is not, and one can expect non-linear effects to be important in determining the detailed behaviour. However, we should still expect to get the effect of a sudden change in the major-axis curve for, as we see from (30), the β_m become large for $m \geqslant N$ when λ_2/λ_1 approaches the integral value N. The numerical integrations indeed show such sudden changes with the major-axis curve changing from one side to another of the curve through C for $\lambda_2/\lambda_1 = 2$ and $\lambda_2/\lambda_1 = 3$. The results are included in figure 3, which is a plot of the (γ, n)-plane showing the different types of behaviour. The dotted lines are the lines along which λ_2/λ_1 is an integer, the line $\lambda_2/\lambda_1 = 1$ marking the change of E from node to spiral. The major-axis curve from E is above or below the curve from C in the different regions of the (γ, n)-plane as shown. The only smooth transitions, and therefore regular solutions, found are those given in table 2, and these lie on the unbroken line in the figure. Presumably, in the particular case of $\gamma = 1{\cdot}75$ where the smooth transition curve intersects the $\lambda_2/\lambda_1 = 2$ curve, the resulting solution must be rejected since it is probably not regular at E.

For all other cases we have a satisfactory solution and I have verified that all the curves go into O in the (Y, Z)-plane after crossing the critical parabola.

It is not yet certain whether all the possible solutions with accelerating fronts have been found. The reason for this is that the survey of the region of interest of the (γ, n)-plane is not complete. A complete mapping of the behaviour of the major-axis solutions for the whole of the (γ, n)-plane that is of interest would

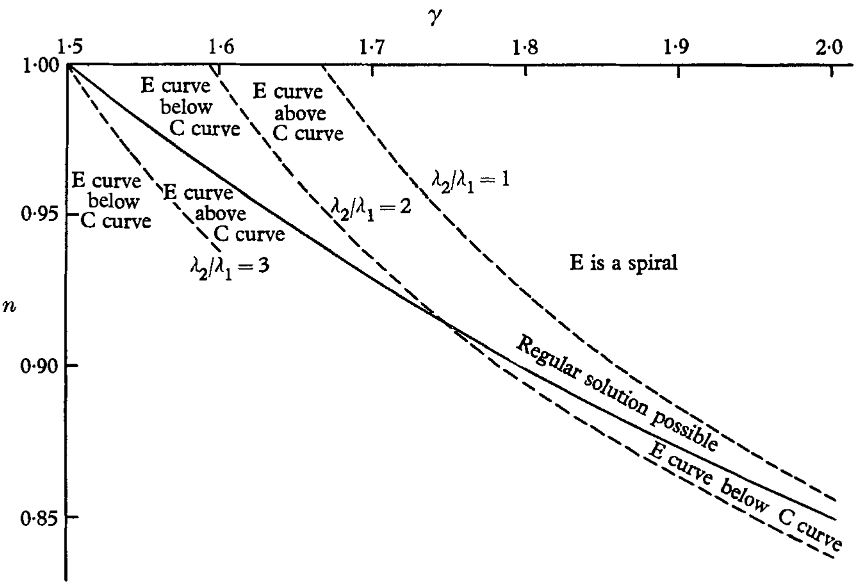

FIGURE 3. The behaviour of the major-axis solution through E and the solution through C in regions of the (γ, n)-plane.

probably be lengthy and tedious. The reason for this is that the ratio λ_2/λ_1 varies from the value 1, when E changes to a spiral, to infinity at the lower end of the range of values of n for which E is a node. There are therefore an infinite number of lines in the (γ, n)-plane on which the ratio λ_2/λ_1 is a positive integer, on each of which the major-axis curve may change sharply though this change need not necessarily be such that the major-axis curve changes sides of the curve from C. Also, in a numerical analysis, a sudden change in a higher-order term of the power series (28) would probably go unnoticed by a finite difference approximation. Another complication is that all the integral curves through E have contact of a high order with the major-axis curve when λ_2/λ_1 is large, as is shown by equation (26). It might therefore be difficult to distinguish numerically between the major-axis curve, and some other curve which starts close to it near E, but later diverges markedly from it. In other words, integration out from E may be highly unstable.

The above remarks apply to all the values of γ that have been considered. We are therefore left with the result that the existence of a series of similarity solutions with accelerating fronts has been established for the range $\frac{3}{2} < \gamma \leqslant 7$, though it is not known whether these are the only such similarity solutions.

5. Collapses at constant velocity

We now consider the special case $n = 1$ in which the cavity collapses with a constant velocity. The problem can be discussed in the (Y, Z)-plane as before, and there are certain simplifications. The (Y, Z)-equation (21) reduces to

$$\frac{dY}{dZ} = \frac{Y[3Z-(Y-1)^2]}{2Z[Z-(Y-1)(\gamma Y-1)]}, \tag{39}$$

which has only four singular points. The origin O and C at $Y = 1$, $Z = 0$ are singular points as before. The others are B at

$$Y = 2/(3\gamma-1), \quad Z = 3(\gamma-1)^2/(3\gamma-1)^2$$

and D at

$$Y = 0, \quad Z = 1.$$

The reason for this reduction from six to four singular points is due to the fact that both E and A have merged with C. The point C is no longer a simple saddle-point, but has a more complicated form as we shall show presently. As before, we need a solution going from C to O. The argument given in § 2 as to why the solution must pass through O has to be modified. We now deduce in a similar way that F and G must be finite at $\xi = 0$, and so Y and Z must vanish there.

To investigate the form of the solution curves near C, we write $Y = 1+y$. The approximate form of (39) near C is

$$\frac{dy}{dZ} = \frac{3Z-y^2}{2Z[Z-(\gamma-1)y]}. \tag{40}$$

From this equation, the form of the integral curves near C can be sketched as in figure 4. The only curves which emerge from C lie below the parabola $3Z = y^2$, and we can further approximate equation (39) for this region by

$$\frac{dy}{dZ} = \frac{y^2-3Z}{2(\gamma-1)yZ}. \tag{41}$$

This equation is linear in y^2 and can be integrated to give

$$\left.\begin{aligned} y^2 &= \frac{3Z}{2-\gamma} + KZ^{1/(\gamma-1)} \quad \text{for} \quad \gamma \neq 2, \\ y^2 &= -3Z\log Z + KZ \quad \text{for} \quad \gamma = 2, \end{aligned}\right\} \tag{42}$$

where K is a constant of integration.

Two different types of behaviour can be distinguished. For $1 < \gamma < 2$, the exponent $1/(\gamma-1)$ is greater than 1. The curves given by (42) therefore consist of the parabola $y^2 = 3Z/(2-\gamma)$ given by the value $K = 0$, and a family of curves which are asymptotic to this parabola as C is approached.

For $\gamma > 2$, the approximate form of the solution curves near C is given by $Z = K^{1-\gamma}y^{2\gamma-2}$, so that all the curves are asymptotic to the y-axis at C. In the intermediate case $\gamma = 2$, the curves are all asymptotic to the curve

$$y^2 = -3Z\log Z.$$

An important fact which emerges from the above investigation is that the solution curves which go through C lie below the critical parabola, and so do not have to cross it to reach O. This is due to the fact that the crossing point E has merged with C. There is now no curve of the type $\xi =$ const. other than the cavity surface which is a characteristic in the (r, t)-plane. We therefore avoid the matching problem we had previously.

It is clear that some of the curves which leave C go into O, and some curl upwards away from O as shown in figure 4. The general form of the integral curves in the (Y, Z)-plane is the same as that shown in Courant & Friedrichs (1948, p. 426, fig. 11). Comparing notation, the present Z is their C^2 and Y is U. They label the singular points C and O with B and A respectively.

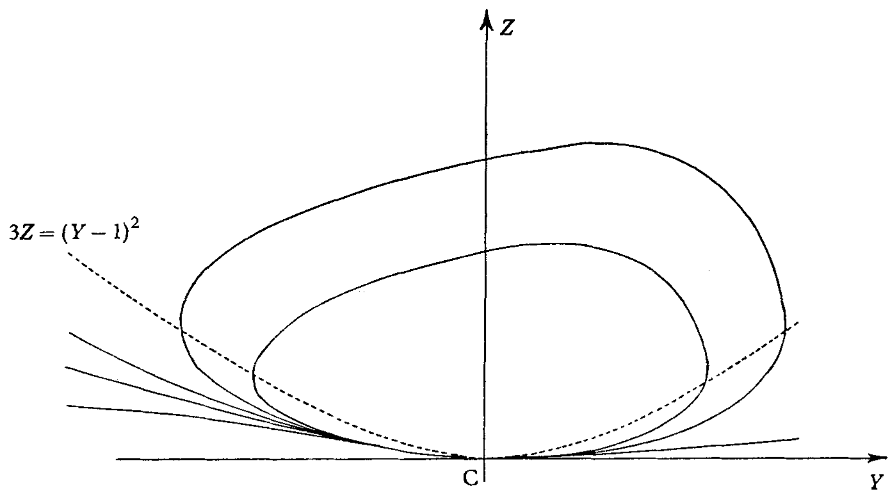

FIGURE 4. The integral curves in the neighbourhood of C when $n = 1$.

Another point to be made is that most of the solutions we have just discussed do not represent functions $Y = Y(\xi)$ and $Z = Z(\xi)$ which are regular at the cavity wall $\xi = -1$. It is not clear whether we should expect these functions to be regular at $\xi = -1$. No problem arose earlier in this respect for $n \neq 1$, because C was then a simple saddle point, and the only non-trivial solution through it was regular there. Although the cavity surface can be regarded as a characteristic since the velocity of sound vanishes there, it does not follow that a singularity at the cavity surface persists in the flow for all time and must be fed in initially. This was the argument we used when we were discussing the so-called limiting characteristic $\xi = \xi_0$ earlier. The standard proof of this latter result on propagation along characteristics does not apply when the characteristic in question is a vacuum front. Gas fronts advancing into vacuum are an individual phenomenon and their properties have not been fully explored.

If one does demand that the similarity solution should have a regular expansion about $\xi = -1$, the number of acceptable similarity solutions is limited. A search for regular expansions of $Y(\xi)$ and $Z(\xi)$ about $\xi = -1$, or, equivalently, of a regular expansion $Z = Z(y)$ of equation (39), shows that such an expansion is possible only for $1 < \gamma < 2$. The series obtained is

$$Z = \frac{(2-\gamma)\,y^2}{3} + \frac{2(\gamma+1)\,(2-\gamma)}{9(3\gamma-5)}\,y^3 + a_4 y^4 + a_5 y^5 + \ldots, \tag{43}$$

where the general equation for a_n $(n \geqslant 3)$ is

$$a_n(n+2-n\gamma) = \text{terms involving } a_2, \ldots, a_{n-1}. \tag{44}$$

A regular expansion therefore is possible provided $\gamma \neq (n+2)/n$ for some integer $n \geqslant 3$. When n does take any one of this sequence of values, no solution is regular at C except the line $Z = 0$.

We should expect the series solution (43) to behave in a similar manner to the major-axis solution, and undergo sudden changes when γ changes through one of the values $(n+2)/n$. This is found to be the case. Numerical integrations to determine the path of this regular solution, using the first four terms of series (43) to start the integration, show that the regular solution goes to O for $2 > \gamma > \frac{5}{3}$, curls upwards away from O for $\frac{5}{3} > \gamma > \frac{3}{2}$, and then goes to O again for $\frac{3}{2} > \gamma > \frac{7}{5}$. As before, the regular curve is a satisfactory solution only when it goes to O. In all cases there are solutions which asymptote to the regular curve at C and which go to O, even when the regular curve at C does not itself go to O.

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